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# A Solvable Six-Vertex Model with Defects

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The general six-vertex model with defective (missing) horizontal bonds leads to a 22-vertex model. A special case, which is isomorphic to a ten-vertex model amendable to the quantum inverse scattering method, is solved in closed form.

**KEY WORDS:** Free energy; vertex model; Bethe Ansatz; inverse scattering method; transfer matrix; ice condition.

Several years ago I had occasion to examine a number of ice models with impurities and defects (in the form of missing bonds). One of these is isomorphic to a class of ten-vertex models[(1)] which has recently been shown to be exactly solvable by the quantum inverse scattering method.[(2)] Both for its intrinsic interest and the light it sheds on Onody and Karowski's model it seems worthwhile to present the closed-form solution.

Consider a general ice model which allows for the absence of horizontal bonds. This leads to the 22-vertex configurations in Fig. 1, with their weight $\alpha_j = \exp(-\beta\epsilon_j)$.

If we set $\alpha_{11} = \cdots = \alpha_{22} = 0$, $\alpha_1 = \alpha_2 = A$, $\alpha_3 = \alpha_4 = B$, $\alpha_0 = \alpha_{10} = AB$ and $\alpha_5 = \cdots = \alpha_8 = C$, we obtain Onody and Karowski's ten-vertex model, which is solvable by the quantum inverse scattering method under the condition $C = [(A + B)(AB - 1)]^{1/2}$. Following Ref. 1, the elementary vertex can be written

$$L_n = \begin{pmatrix} a_n & b_n \\ b_n^+ & c_n \end{pmatrix}$$

where the subscript indicates that the operator affects the $n$th position in a

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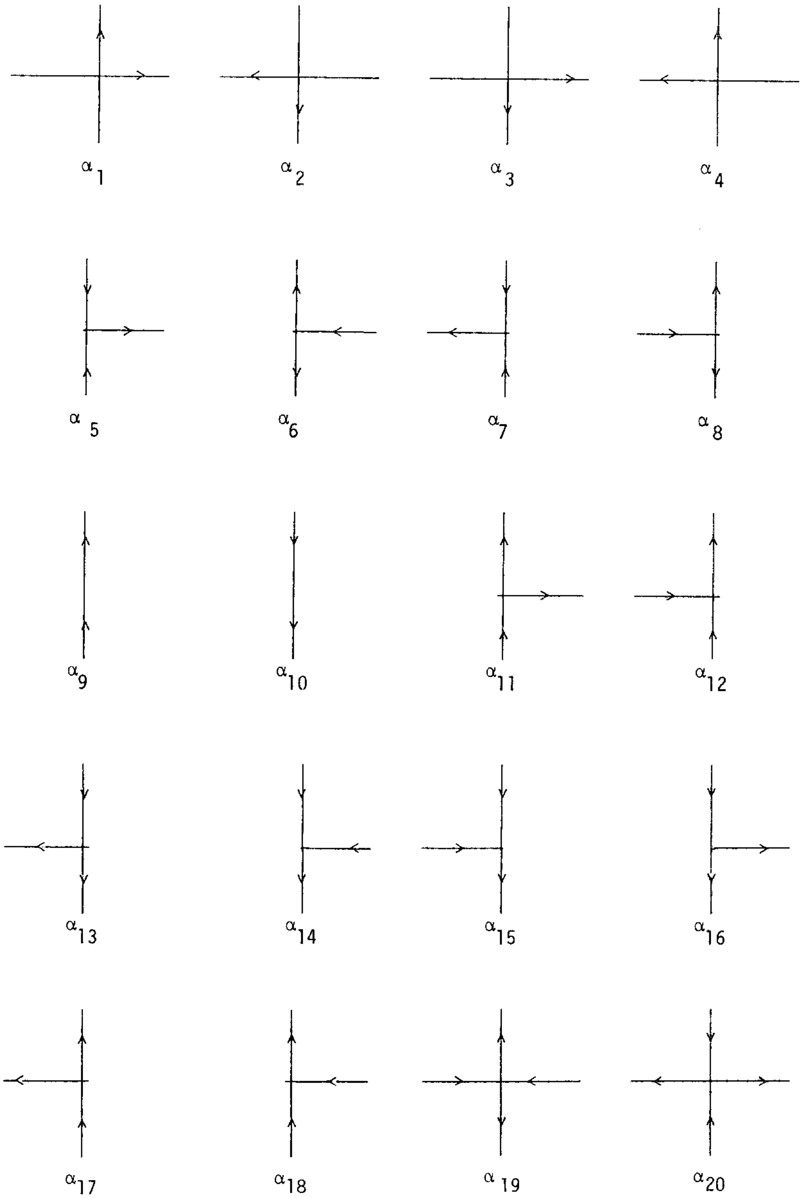


Fig. 1. The 22-vertex configurations corresponding to an "ice" model with defective horizontal bonds. The vertex weights are $\alpha_j = \exp(-\beta\epsilon_j)$, $\beta = 1/k_B T$. (Configurations $\alpha_{21}\alpha_{22}$, consisting of opposite vertical arrows, are not shown.)

row (we consider an $N \times N$ lattice with periodic boundary conditions), and

$$a = \mathrm{diag}(A, AB, B), \qquad c = \mathrm{diag}(B, AB, A), \qquad b = \begin{bmatrix} 0 & 0 & 0 \\ c & 0 & 0 \\ 0 & c & 0 \end{bmatrix}$$

Then the monodromy matrix, whose trace is the transfer matrix, is

$$\prod_{n=1}^{N} L_n = \begin{pmatrix} X(\nu), & Y(\nu) \\ Z(\nu), & W(\nu) \end{pmatrix}$$

and the entries satisfy the commutation relation

$$[Y(\nu), Y(\nu')] = 0$$

$$X(\nu)Y(\nu') = \frac{\sin(\eta + \nu' - \nu)}{\sin(\nu - \nu')} Y(\nu')X(\nu) - \frac{\sin\eta}{\sin(\nu - \nu')} Y(\nu)X(\nu')$$

$$W(\nu)Y(\nu') = \frac{\sin(\eta + \nu - \nu')}{\sin(\nu - \nu')} Y(\nu')W(\nu) + \frac{\sin\eta}{\sin(\nu - \nu')} Y(\nu)W(\nu')$$

where

$$\begin{aligned} \tan\nu &= (A + B)^2/(B - A)\left[4A^2B^2 - (A + B)^2\right]^{1/2} \\ \cos\eta &= -(A + B)/2AB \end{aligned} \tag{1}$$

This corresponds to Onody and Karowski's parameterization. By quantum inverse scattering[2] this leads to an eigenvalue condition [Eq. (10) in Ref. 1], which in the thermodynamic limit can be cast as an integral equation. The solution, by the methods of Ref. 3, gives the free energy

$$-\beta F = \log\left|\tan\frac{\pi}{2\eta}\left(\frac{\pi}{2} + \nu\right)\tan\frac{\pi}{2\eta}\left(\frac{\pi}{2} - \nu\right)\right| + 2\begin{cases} \ln A, & \nu > 0 \\ \ln B, & \nu < 0 \end{cases} \tag{2}$$

Onody and Karowski(1) have discussed the phase diagram of the model in the $A$–$B$ plane. Equations (1) and (2) allow the critical indices to be found by elementary analytic methods.

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